$$+2a^2\int\sqrt{(r^2-a^2\sin^2\theta)}d\theta-2a^2\int_0^{\frac{1}{2}\pi}\sin^2\theta\sqrt{(r^2-a^2\sin^2\theta)}d\theta.$$

When $\theta=0$, the first line is $\frac{1}{2}(\pi a^2 r)$.
When $\theta=\frac{1}{2}\pi$, its value is 0.

To evaluate $\int\sqrt{(r^2-a^2\sin^2\theta)}d\theta$, put $a=re$, expand by binomial formula, and integrate the terms.

$$(1-e^2\sin^2\theta)^{\frac{1}{2}}=1-\tfrac{1}{2}e^2\sin^2\theta-\tfrac{1}{8}e^4\sin^4\theta-\tfrac{1}{16}e^6\sin^6\theta-\ldots\ldots$$

Then $\int_0^{\frac{1}{2}\pi}\sqrt{(r^2-a^2\sin^2\theta)}d\theta=\frac{1}{2}(\pi r)(1-\frac{1}{4}e^2-\frac{3}{64}e^4-\frac{5}{256}e^6\ldots\ldots)$

The remaining integral treated in the same way gives

$$\tfrac{1}{2}\pi r(\tfrac{1}{2}-\tfrac{3}{16}e^2-\tfrac{5}{128}e^4-\tfrac{35}{2048}e^6-\ldots\ldots$$

Finally, Volume$=\dfrac{\pi a^2 r}{8}\left(e^2+\frac{1}{8}e^4+\dfrac{5e^6}{128}+\ \ldots\right)$

In our example, $e=\frac{1}{6}$, $a=1$, $r=6$.

Hence, $V=\dfrac{9\pi}{2}\left(\dfrac{1}{36}+\dfrac{1}{8\times1296}+\dfrac{5}{128\times6^6}+\ldots\ldots=\frac{1}{8}\pi(1+\frac{1}{288})\right.$ nearly.

This subtracted from $\pi a^2 r$ and the result doubled gives the volume common to the two cylinders.

MECHANICS.

90. Proposed by **WALTER H. DRANE**, Graduate Student, Harvard University, Cambridge, Mass.

Adopting the hypothesis that the planets were originally all one mass revolving about a fixed center and were formed by an explosion of this mass at some point in its path; prove that, if the law of nature were that force varies directly as the distance, the planets would all have collided again simultaneously, and find an expression for the time between the explosion and collision.

Solution by the PROPOSER.

Regarding the original mass as a particle, the pieces after explosion, no matter what their initial velocities or directions, would all move in concentric ellipses; and as their paths intersect in one point, viz., the position of the original mass at the moment of explosion, they must all have another point in common at the extremity of the common diameter through the first common point.

We have for the equations of motion for any piece

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=0 \;\ldots\ldots(1). \qquad m\left[\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2\right]=-r\ldots\ldots(2).$$

From (1) by integration,

$$r^2\frac{d\theta}{dt}=h\ldots\ldots(3),$$

a constant depending on the initial velocity and angle of projection.

To integrate (2) let $u=1/r$. Then

$$\frac{dr}{dt}=-\frac{1}{u^2}\frac{du}{dt}=-\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}=-h\frac{du}{dt} \text{ from (3).}$$

$$\therefore \frac{d^2r}{dt^2}=-h\frac{d}{dt}\left(\frac{du}{d\theta}\right)=-h^2u^2\frac{d^2u}{d\theta^2}.$$

Substituting these results in (2) and using (3) we have

$$\frac{d^2u}{d\theta^2}=-u+\frac{k}{h^2u^3}\ldots\ldots(4),$$

where k is a constant depending on the force of attraction.

Multiply by $2\frac{du}{d\theta}$ and integrate and we have

$$\left(\frac{du}{d\theta}\right)^2=-u^2-\frac{k}{h^2u^2}+c_1\ldots\ldots(5),$$

$$\theta=\int\frac{hudu}{\sqrt{(c_1h^2u^2-h^2u^4-k)}}=\tfrac{1}{2}\cos^{-1}\left(\frac{2hu^2-c_1h}{\sqrt{(c_1^2h_2-4k)}}\right)+c_2.$$

Hence simplifying and restoring value of u we get for the equation of the path

$$\frac{1}{r^2}=\frac{c_1}{2}-\frac{\sqrt{(c_1^2h^2-4k)}}{2h}+\frac{\sqrt{(c_1^2h^2-4k)}}{h}\cos^2(\theta-c_2).$$

Transforming to rectangular coödinates this becomes

$$\frac{x^2}{\dfrac{2h}{c_1h+\sqrt{(c_1^2h^2-4k)}}}+\frac{y^2}{\dfrac{2h}{c_1h-\sqrt{(c_1^2h^2-4k)}}}=1.$$

Let A be area of the ellipse, and a, b, its semi-axes. Then

$$A=\pi ab=\pi\times\frac{2h}{c_1h+\sqrt{(c_1^2h_2-4k)}}\times\frac{2h}{c_1h-\sqrt{(c_1^2h_2-4k)}}=\frac{\pi h}{\sqrt{k}}.$$

For the area of any curve, we have from the calculus

$$A=\tfrac{1}{2}\int_{\theta_0}^{\theta_1}r^2d\theta.$$

If we regard the angle θ_0 as the initial angle it is 0, and as θ is a function of the time, this integral may be written

$$A=\tfrac{1}{2}\int_0^{t_1}r^2d\theta=\tfrac{1}{2}\int_0^t r^2\frac{d\theta}{dt}dt=\tfrac{1}{2}h\int_0^{t_1}dt=\tfrac{1}{2}ht \text{ by } (3).$$

$\therefore$ $t=2A/h$. That is, the periodic time always equals twice the area swept over by the radius vector divided by the constant h.

Hence for the given ellipse we have

$$t=\frac{2\pi h/\sqrt{k}}{h}=\frac{2\pi}{\sqrt{k}}.$$

Therefore the time it would require for a piece to travel from the point of explosion to the next point of intersection is $\tfrac{1}{2}t-\pi/\sqrt{k}$, and as k is a constant the same for all pieces, we see this time is the same for all, and hence they must collide simultaneously.

91. Proposed by CHARLES C. CROSS, Whaleyville, Va.

The bow of a boat which is a inches wide is inclined at an angle α. When in motion in perfectly calm water the water was found to rise b inches on the bow. Required the velocity of the boat.

No solution of this problem has been received.

92. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

A particle, starting at the vertex, slides down a smooth parabolic curve. Find the initial velocity of the particle so that it may leave the curve at the extremity of semi-latus rectum.

Solution by GEORGE R. DEAN, A. M., Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

Take the vertex as origin and positive x downwards.

The equation of the curve is $y^2=2px$.

Let N be the normal reaction, v the velocity, θ the angle between normal and x-axis, m the mass of particle, ρ the radius of curvature.

Then $N=mv^2/\rho+mg\cos\theta$.

$$\rho=\frac{(y^2+p^2)^{\frac{3}{2}}}{p^2},\quad v^2=v_0^2+2gx,\quad \cos\theta=\frac{p}{\sqrt{(y^2+p^2)}}.$$